MATH 1650: SECTION 2.3: REAL ZEROS OF POLYNOMIALS

In this section, we formulate a general strategy for finding the real zeros of a polynomial function. First things first, how many real zeros can a polynomial function have?

THE n ZEROS THEOREM Counting multiplicities, a polynomial of degree n has at most n real zeros.

So, for example, $f(x) = 2x^4 + 4x^3 - x^2 - 6x - 3$ is degree 4, so f has at most 4 real zeros, counting multiplicities. (Can you see how this connects with the Factor Theorem?)

Now that we know how many zeros to look for, we need to formulate a strategy to find them.

STRATEGY FOR FINDING REAL ZEROS OF POLYNOMIAL FUNCTIONS:

1. Use tools to help make an educated guess as to what number is a zero.
2. Use synthetic division to check our guess.
3. Rinse and repeat until we are left with (at worst) a quadratic factor.

Of course, one tool we'll be using is a graphing utility (like desmos.) However, remember that graphing utilities will only **suggest** solutions. Ultimately, algebra is the only way we can **prove** our answers. Another tool we will use is the mathematical tool (a **theorem**) stated below.

THE RATIONAL ZEROS THEOREM The **rational** zeros of a polynomial function are among the list:

$$\pm\frac{\text{factors of the constant term}}{\text{factors of the leading coefficient}}$$

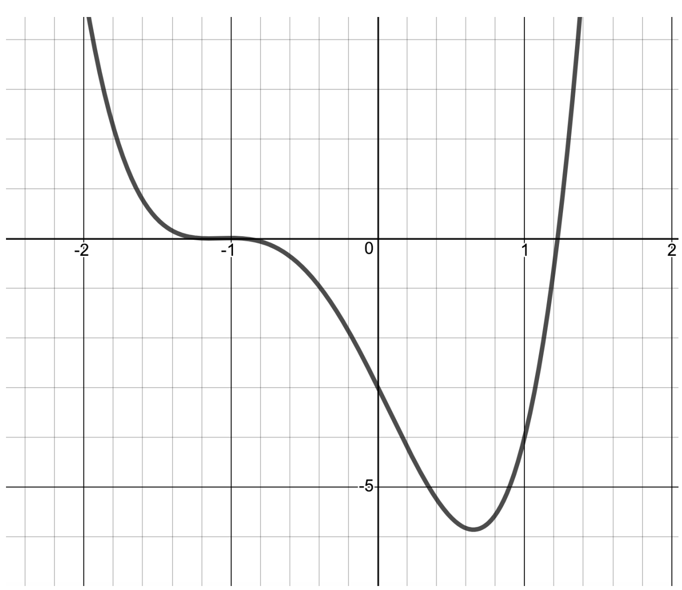
The Rational Zeros Theorem gives us a list of numbers to try in our synthetic division and that is a lot nicer than simply guessing. If none of the numbers in the list are zeros, then either the polynomial has no real zeros at all, or all of the real zeros are irrational numbers. We put this new tool to use in the next example.

EXAMPLE: $f(x) = 2x^4 + 4x^3 - x^2 - 6x - 3$.

- Use the Rational Zeros Theorem to list all of the possible rational zeros of f.

  To generate a complete list of rational zeros, we need to take each of the factors of constant term, $a_0 = -3$, and divide them by each of the factors of the leading coefficient $a_4 = 2$. The factors of -3 are ± 1 and ± 3. Since the Rational Zeros Theorem tacks on a $\pm$ anyway, for the moment, we consider only the positive factors 1 and 3. The factors of 2 are 1 and 2, so the Rational Zeros Theorem gives the list $\left\{\pm\frac{1}{1}, \pm\frac{1}{2}, \pm\frac{3}{1}, \pm\frac{3}{2}\right\}$ or $\left\{\pm\frac{1}{2}, \pm 1, \pm\frac{3}{2}, \pm 3\right\}$.

- Graph $y = f(x)$ using a graphing utility.



- Use the graph to shorten the list of possible rational zeros.

  Of the numbers in the list $\left\{\pm\frac{1}{2}, \pm 1, \pm\frac{3}{2}, \pm 3\right\}$, only $x = -1$ looks like it may be a zero. In fact, the graph **appears** to cross the x-axis with an 's'-shape at $(-1, 0)$ so $x = -1$ could be a zero of multiplicity 3.

- Use synthetic division to find the real zeros of f, and state their multiplicities.

  We start dividing $x = -1$ into $f(x)$:

$$\begin{array}{r|rrrrr} -1 & 2 & 4 & -1 & -6 & -3 \\ & \downarrow & -2 & -2 & 3 & 3 \\ \hline & 2 & 2 & -3 & -3 & \boxed{0} \end{array}$$

Since f is a fourth degree polynomial, we know that our quotient is a third degree polynomial. If we can do one more successful division, we will have reduced the quotient to a quadratic, and we can use the quadratic formula, if needed, to find the two remaining zeros. Continuing with $x = -1$:

$$\begin{array}{r|rrrrr} -1 & 2 & 4 & -1 & -6 & -3 \\ & \downarrow & -2 & -2 & 3 & 3 \\ \hline -1 & 2 & 2 & -3 & -3 & \boxed{0} \\ & \downarrow & -2 & 0 & 3 & \\ \hline & 2 & 0 & -3 & \boxed{0} & \end{array}$$

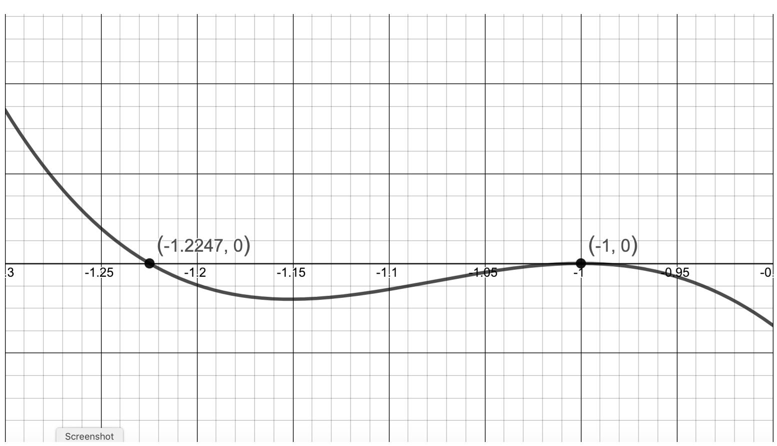
Our quotient polynomial is now $2x^2 - 3$. Solving $2x^2 - 3 = 0$ we get $x^2 = \frac{3}{2}$, so $x = \pm\sqrt{\frac{3}{2}} = \pm\frac{\sqrt{6}}{2}$.

Since f is a 4th degree polynomial, we know to expect 4 zeros, counting multiplicity.

We divided $x = -1$ **twice**, and then obtained **two** other zeros, $x = \pm\frac{\sqrt{6}}{2}$, so we have our total of 4 zeros.

There is an important lesson here about the limits of technology. Looking at the initial graph of $y = f(x)$, it certainly **appeared** as if $x = -1$ were a zero of multiplicity three. However we now know thanks to synthetic division that $x = -1$ is a zero of multiplicity **two** only.

If we zoom in near $(-1, 0)$ using graphing utility, we find the graph of $y = f(x)$ touches and rebounds from the x-axis at $(-1, 0)$, typical behavior near a zero of multiplicity two. Remember, **technology** may **suggest** a result, but it is only the **mathematics** which can **prove** (or in this case, **disprove**) it.



EXAMPLE: Find all real zeros of the following polynomial functions using the following procedure:

- Use the Rational Zeros Theorem (RZT) to list all of the possible rational zeros.
- Graph the function using a graphing utility.
- Use the graph to shorten the list of possible rational zeros.
- Use synthetic division to find the real zeros of the function, and state their multiplicities.

1. $p(x) = 2x^3 - 5x^2 - 4x + 12$

2. $q(z) = 3z^3 - 10z^2 + 7z + 10$

EXAMPLE: Solve the following equations and/or inequalities.

- $4x^5 + 8x^4 + 5x^3 = 5x^2 + 4x - 2$

  HINT: Rewrite the equation in the form $f(x) = 0$ and find the real zeros of f.

- $x^4 + 12x^2 + 1 \geq 7x(x^2 + 1)$

  HINT: Rewrite the inequality in the form $f(x) \geq 0$ and make a Sign Diagram.

HOMEWORK: Section 2.3: 11 - 29 odd, 35 - 53 odd